Triple Integrals



Goal: Integrate functions of three variables over solids...

↳ Fun Fact: This works exactly as before.



Idea: Transform an integral over a solid region into an iterated integral by parametrizing the region nicely. ☆

NB: All the properties of double integrals extend to this new setting in the most obvious ways possible...

Notation: The (triple) integral of $f : D \subseteq \mathbb{R}^3 \to \mathbb{R}$ over $R \subseteq D$ is denoted $\iiint_R f(x,y,z)\, dV$.

The best way to get familliar with triple integrals at this level is to jump straight into some examples.

Ex: Compute $\iiint_R (xy+z^2)\,dV$ for $R=[0,2]\times[0,1]\times[0,3]$.

$\quad = \left\{(x,y,z) : \begin{matrix} 0\le x\le 2, \\ 0\le y\le 1, \\ 0\le z\le 3 \end{matrix}\right\}$

Sol: $\iiint_R (xy+z^2)\,dV$

$$= \int_{x=0}^{2}\int_{y=0}^{1}\int_{z=0}^{3} (xy+z^2)\,dz\,dy\,dx$$

Step 1: $\int_{z=0}^{3} (xy+z^2)\,dz = \left[xyz + \frac{1}{3}z^3\right]_{z=0}^{3}$

$$= \left(3xy + \frac{27}{3}\right) - 0 = 3xy + 9$$

Step 2: $\int_{y=0}^{1} (3xy+9)\,dy = 3\int_{y=0}^{1} (xy+3)\,dy$

$$= 3\left[\frac{1}{2}xy^2 + 3y\right]_{y=0}^{1} = 3\left(\frac{1}{2}x + 3 - 0\right) = \frac{3}{2}x + 9$$

Step 3: $\int_{x=0}^{2} \left(\frac{3}{2}x + 9\right) dx = \left[\frac{3}{4}x^2 + 9x\right]_{x=0}^{2} = 3\cdot\frac{4}{4} + 9\cdot 2 - 0$

$$= 3 + 18 = 21$$

$\therefore \iiint_R (xy+z^2)\,dV = 21.$ $\boxed{//}$

Ex: Compute $\iiint_R (2x-y)\,dV$ for $R = \{(x,y,z) : 0 \le z \le 2,\ 0 \le y \le z^2, \text{ and } 0 \le x \le y-z\}$

Sol:

$$\iiint_R (2x-y)\,dV = \int_{z=0}^{2} \int_{y=0}^{z^2} \int_{x=0}^{y-z} (2x-y)\,dx\,dy\,dz$$

Step 1: $\int_{x=0}^{y-z} (2x-y)\,dx = \left[x^2 - xy\right]_{x=0}^{y-z}$

$= \left((y-z)^2 - y(y-z)\right) - 0 = y^2 - 2yz + z^2 - y^2 + yz$

$= z^2 - yz$

Step 2: $\int_{y=0}^{z^2} (z^2 - yz)\,dy = \left[yz^2 - \frac{1}{2}y^2 z\right]_{y=0}^{z^2}$

$= \left(z^2 \cdot z^2 - \frac{1}{2}(z^2)^2 \cdot z\right) - 0 = z^4 - \frac{1}{2}z^5$

Step 3: $\int_{z=0}^{2} \left(z^4 - \frac{1}{2}z^5\right)dz$

$= \left[\frac{1}{5}z^5 - \frac{1}{12}z^6\right]_{z=0}^{2} = \left(\frac{1}{5}\cdot 32 - \frac{1}{12}\cdot 64\right) - 0$

$= \frac{32}{5} - \frac{16}{3} = \frac{96-80}{15} = \frac{16}{15}$

$\therefore \iiint_R (2x-y)\,dV = \frac{16}{15}$ ■

Ex: Compute $\iiint_R y \, dV$ for $R = \{(x,y,z) : 0 \le x \le 3,\ 0 \le y \le x, \text{ and } x-y \le z \le x+y\}$.

Sol: $$\iiint_R y \, dV = \int_{x=0}^{3} \int_{y=0}^{x} \int_{z=x-y}^{x+y} y \, dz \, dy \, dx$$

Step 1: $$\int_{z=x-y}^{x+y} y \, dz = y \left[z \right]_{z=x-y}^{x+y}$$

$$= y\left((x+y) - (x-y) \right) = 2y^2$$

Step 2: $$\int_{y=0}^{x} 2y^2 \, dy = \frac{2}{3} \left[y^3 \right]_{y=0}^{x} = \frac{2}{3}\left(x^3 - 0 \right) = \frac{2}{3} x^3$$

Step 3: $$\int_{x=0}^{3} \frac{2}{3} x^3 \, dx = \frac{2}{3} \cdot \frac{1}{4} \left[x^4 \right]_{x=0}^{3} = \frac{1}{2 \cdot 3}\left(3^4 - 0 \right)$$

$$= \frac{27 \cdot 3}{2 \cdot 3} = \frac{27}{2}$$

$$\therefore \iiint_R y \, dV = \frac{27}{2}$$

$\blacksquare$

Ex: Compute $\iiint_R x e^{-y}\, dV$ for $R = \left\{ (x,y,z) : \begin{array}{l} 1 \le z \le 2, \quad 1 \le x \le 2z, \\ \text{and} \quad 0 \le y \le \ln(x) \end{array} \right\}$

Sol: $$\iiint_R x e^{-y}\, dV = \int_{z=1}^{2} \int_{x=1}^{2z} \int_{y=0}^{\ln(x)} x e^{-y}\, dy\, dx\, dz$$

Step 1: $$\int_{y=0}^{\ln(x)} x e^{-y}\, dy = -x \left[e^{-y} \right]_{y=0}^{\ln(x)}$$

$$= -x\left(e^{-\ln(x)} - e^{0}\right) = -x\left(e^{\ln(x^{-1})} - 1\right)$$

$$= -x\left(x^{-1} - 1\right) = -1 + x = x - 1$$

Step 2: $$\int_{x=1}^{2z} (x-1)\, dx = \left[\frac{1}{2}x^2 - x\right]_{x=1}^{2z}$$

$$= \left(\frac{1}{2}(2z)^2 - 2z\right) - \left(\frac{1}{2} - 1\right)$$

$$= 2z^2 - 2z + \frac{1}{2}$$

Step 3: $$\int_{z=1}^{2} \left(2z^2 - 2z + \frac{1}{2}\right) dz = \left[\frac{2}{3}z^3 - z^2 + \frac{1}{2}z\right]_{z=1}^{2}$$

$$= \left(\frac{2}{3}\cdot 8 - 4 + 1\right) - \left(\frac{2}{3} - 1 + \frac{1}{2}\right)$$

$$= \frac{16}{3} - 3 - \frac{2}{3} + 1 - \frac{1}{2}$$

$$= \frac{14}{3} - \frac{5}{2} = \frac{28-15}{6} = \frac{13}{6}$$

$$\therefore \iiint_R x e^{-y}\, dV = \frac{13}{6}$$

$\boxed{\checkmark}$

Ex: Compute $\iiint_R e^{z/y}\,dV$ for $R = \left\{(x,y,z): \begin{array}{l} 0 \le y \le 1,\ y \le x \le 1, \\ \text{and } 0 \le z \le xy \end{array}\right\}$

Sol:
$$\iiint_R e^{z/y}\,dV = \int_{y=0}^{1}\int_{x=y}^{1}\int_{z=0}^{xy} e^{z/y}\,dz\,dx\,dy$$

Step 1: $\int_{z=0}^{xy} e^{z/y}\,dz = \int_{z=0}^{xy} y\,e^{z/y}\cdot\frac{1}{y}\,dy$ $\left(\begin{array}{l} u = z/y \\ du = \frac{1}{y}\,dy \end{array} \quad \begin{array}{l} u(xy) = x \\ u(0) = 0 \end{array}\right)$

$$= \int_{u=0}^{x} y e^{u}\,du = y\left[e^{u}\right]_{u=0}^{x} = y(e^{x}-1)$$

Step 2: $\int_{x=y}^{1} y(e^{x}-1)\,dx = y\left[e^{x} - x\right]_{x=y}^{1}$

$$= y\left((e^{1}-1)-(e^{y}-y)\right) = y(e-1-e^{y}+y)$$

$$= y^{2} + (e-1)y - ye^{y}$$

Step 3: $\int_{y=0}^{1}\left(y^{2} + (e-1)y - ye^{y}\right)dy$

$$= \int_{y=0}^{1}\left(y^{2}+(e-1)y\right)dy - \int_{y=0}^{1} ye^{y}\,dy$$ $\left(\begin{array}{l} u = y \\ du = dy \end{array} \quad \begin{array}{l} dv = e^{y}dy \\ v = e^{y} \end{array}\right)$

$$= \left[\frac{1}{3}y^{3} + \frac{(e-1)}{2}y\right]_{y=0}^{1} - \left[ye^{y} - \int e^{y}\,dy\right]_{y=0}^{1}$$

$$= \left(\left(\frac{1}{3}+\frac{e-1}{2}\right)-0\right) - \left[ye^{y}-e^{y}\right]_{y=0}^{1}$$

$$= \frac{1}{3} + \frac{e-1}{2} - \left((e^{1}-e^{1})-(0-1)\right)$$

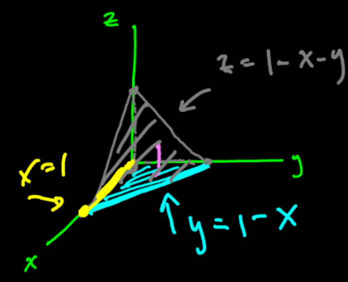
$$= \frac{1}{3} + \frac{e-1}{2} - (0+1) = -\frac{2}{3} + \frac{e-1}{2} = \frac{-4+3e-3}{6} = \frac{3e-7}{6}$$

$$\therefore \iiint_R e^{z/y}\,dV = \frac{3e-7}{6}$$ ◪

Ex: Compute the volume of the tetrahedron having vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$.

Sol: $\vec{u} = \langle 1, 0, -1 \rangle \quad \vec{v} = \langle 1, -1, 0 \rangle$

PICTURE



$$\vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{vmatrix} = \langle -1, -(0+1), -1 \rangle = \langle -1, -1, -1 \rangle$$

$\therefore$ Plane passing through $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ has equation

$$0 = 1(x-1) + 1(y-0) + 1(z-0) = x - 1 + y + z$$

i.e. $z = 1 - x - y$ $\quad\therefore$ z-bounds are: $0 \leq z \leq 1 - x - y$.

Hence our region $R = \{(x,y,z) : 0 \leq x \leq 1,\ 0 \leq y \leq 1-x,\ 0 \leq z \leq 1-x-y\}$

$$\therefore \text{Vol}(R) = \iiint_R 1 \, dV$$

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} dz\, dy\, dx$$

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} [z]_{z=0}^{1-x-y} dy\, dx = \int_{x=0}^{1} \int_{y=0}^{1-x} (1 - x - y)\, dy\, dx$$

$$= \int_{x=0}^{1} \left[y(1-x) - \tfrac{1}{2} y^2 \right]_{y=0}^{1-x} dx$$

$$= \int_{x=0}^{1} \left((1-x)^2 - \tfrac{1}{2}(1-x)^2 \right) dx = \tfrac{1}{2} \int_{x=0}^{1} \left(1 - 2x + x^2 \right) dx$$

$$= \tfrac{1}{2} \left[x - x^2 + \tfrac{1}{3} x^3 \right]_{x=0}^{1} = \tfrac{1}{2}\left(1 - 1 + \tfrac{1}{3} - 0 \right) = \tfrac{1}{6} \quad \boxed{//}$$

Center of Mass



Suppose solid object $E \subseteq \mathbb{R}^3$ has <u>density</u> $\rho(x,y,z)$ on E.

The <u>mass</u> of E is $m = \iiint_E \rho(x,y,z)\,dV$.

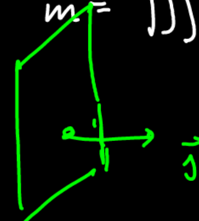
The <u>center of mass</u> of E is the point $(\bar{x}, \bar{y}, \bar{z})$ where every plane through $(\bar{x}, \bar{y}, \bar{z})$ has exactly half of the mass of E on either side.

Q: How can we find $(\bar{x}, \bar{y}, \bar{z})$?

A: Via some hard work <u>(omitted)</u> it is possible to show:

$$\begin{cases} \bar{x} = \frac{1}{m} \iiint_E x\,\rho(x,y,z)\,dV \\ \bar{y} = \frac{1}{m} \iiint_E y\,\rho(x,y,z)\,dV \\ \bar{z} = \frac{1}{m} \iiint_E z\,\rho(x,y,z)\,dV \end{cases}$$

(where $m = \iiint_E \rho(x,y,z)\,dV$).



↳ <u>Idea</u>: These take advantage of the standard basis vectors to "slide" their normal planes into the position at which the mass of E is exactly halved...

Ex: Compute the center of mass of the solid cube $E = \{(x,y,z) : 0 \le x,y,z \le 3\}$ with density $\rho(x,y,z) = x^2+y^2+z^2$.

Sol: Outline: ① mass, ② $\bar{x}$, ③ $\bar{y}$, ④ $\bar{z}$, ⑤ ☑

Mass: $m = \iiint_E \rho(x,y,z)\, dV$

$$= \int_{x=0}^{3} \int_{y=0}^{3} \int_{z=0}^{3} (x^2+y^2+z^2)\, dz\, dy\, dx$$

$$= \int_{x=0}^{3} \int_{y=0}^{3} \left[(x^2+y^2) z + \frac{1}{3} z^3 \right]_{z=0}^{3} dy\, dx$$

$$= \int_{x=0}^{3} \int_{y=0}^{3} \left(3(x^2+y^2) + \frac{1}{3}\cdot 27 - 0 \right) dy\, dx$$

$$= 3 \int_{x=0}^{3} \int_{y=0}^{3} (x^2 + 3 + y^2)\, dy\, dx$$

$$= 3 \int_{x=0}^{3} \left[(x^2+3) y + \frac{1}{3} y^3 \right]_{y=0}^{3} dx$$

$$= 3 \int_{x=0}^{3} \left(\left(3(x^2+3) + \frac{1}{3}\cdot 27 \right) - 0 \right) dx$$

$$= 9 \int_{x=0}^{3} (x^2+6)\, dx = 9 \left[\frac{1}{3} x^3 + 6x \right]_{x=0}^{3}$$

$$= 9 \left(\frac{1}{3}\cdot 27 + 6\cdot 3 - 0 \right) = 9 \cdot (9+18)$$

$$= 9 \cdot 27 = 243$$

Hence the mass is $m = 243$.

$$\bar{x} = \frac{1}{m} \iiint_E x\rho(x,y,z)\, dV$$

$$= \frac{1}{243} \int_{x=0}^{3} \int_{y=0}^{3} \int_{z=0}^{3} x\left(x^2+y^2+z^2\right) dz\, dy\, dx$$

Step 1: $\int_{z=0}^{3} \left(x^3 + xy^2 + xz^2\right) dz = \left[z\left(x^3+xy^2\right) + \frac{1}{3} x z^3 \right]_{z=0}^{3}$

$$= \left(3\left(x^3 + xy^2\right) + \frac{1}{3} \cdot 27x - 0 \right) = 3\left(x^3 + xy^2 + 3x\right)$$

Step 2: $3\int_{y=0}^{3} \left(x^3 + xy^2 + 3x\right) dy = 3\left[y\left(x^3+3x\right) + \frac{1}{3} x y^3 \right]_{y=0}^{3}$

$$= 3\left(3\left(x^3+3x\right) + \frac{1}{3} \cdot 27x - 0 \right)$$

$$= 9\left(x^3 + 3x + 3x \right) = 9\left(x^3 + 6x \right)$$

Step 3: $9\int_{x=0}^{3} \left(x^3+6x\right) dx = 9\left[\frac{1}{4} x^4 + 3x^2 \right]_{x=0}^{3}$

$$= 9\left(\frac{81}{4} + 27 - 0 \right) = 9 \cdot 27 \cdot \left(\frac{3}{4} + 1 \right) = 243 \cdot \frac{7}{4}$$

$$\therefore \bar{x} = \frac{1}{243} \iiint_E x\, \rho(x,y,z)\, dV = \frac{1}{243} \cdot 243 \cdot \frac{7}{4} = \frac{7}{4}$$

NB: By symmetry of E and ρ, we end up with $\bar{x} = \bar{y} = \bar{z}$ in this case.

$\therefore$ the center of mass is $\left(\frac{7}{4}, \frac{7}{4}, \frac{7}{4}\right)$ $\boxed{//}$